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Engineering Notes

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Optimal Rotation Angle About a Nonnominal Euler Axis

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I. Introduction

EULER axis/angle is a classical representation in many attitude determination/control problems [1]. In attitude control problems, for example, this form of attitude representation is very useful to visualize the rotation which takes an “initial” reference frame to a “target” reference frame. In many practical cases, however, a desired rotation about an Euler axis by the relative Euler angle is simply not attainable. This is the case, for example, in underactuated spacecraft (S/C) [2–4], where a certain rotation about a principal rotation axis cannot be carried out because an actuator capable of generating a torque component about that axis is not available (or has stopped working because of a failure). Another example is represented by a magnetically controlled S/C where a desired torque cannot always be generated onboard [5–8]. As a matter of fact, following the cross-product law $\boldsymbol{m} = (\boldsymbol{d} \times \boldsymbol{B})$, the body-referenced attainable torque $\boldsymbol{m}$ is always perpendicular to the body-referenced external magnetic field $\boldsymbol{B}$ (whose direction is, of course, not controllable) and to the onboard-generated magnetic dipole $\boldsymbol{d}$.

Therefore, sometimes it must be accepted that a rotation about a “nonnominal” axis (the “nominal” axis being the one which takes the initial reference frame over the target reference frame through a single rotation) could be performed. The goal of the work presented here is to provide an exact analytical expression to compute the optimal rotation angle about a nonnominal rotation axis, which minimizes the alignment error between the target and the attainable attitude. An immediate application of this work to magnetically controlled S/C can be the design of a three-axis control law in which the instantaneous feasible rotation axis $\boldsymbol{g}$ is derived from the desired rotation axis $\boldsymbol{e}$ using the equation $\boldsymbol{g} = (\boldsymbol{B} \times \boldsymbol{e}/\|\boldsymbol{B} \times \boldsymbol{e}\|) \times \hat{\boldsymbol{B}}$, and the rotation angle about $\boldsymbol{g}$, to be used as the feedback term, is obtained using the method proposed in this note.

The reminder of this note is organized as follows. First some mathematical preliminaries about the Euler axis/angle attitude parameterization are presented and the analytical solution for rotations about a nonnominal axis is developed. In the next section, some numerical results are presented and, finally, some conclusions are drawn about the accuracy and applicability of the proposed procedure.

II. Preliminaries

We define two arbitrary Cartesian coordinate frames: the initial reference frame $\mathbb{F}_1$ and the target reference frame $\mathbb{F}_2$.

Let $\mathbb{T}_{21} \in \Re^{3\times 3}$ represent the rotation matrix between the two frames; this means

$$\boldsymbol{v}_2 = \mathbb{T}_{21}\boldsymbol{v}_1 \tag{1}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ represent a generic vector, as expressed in the initial and target frames, respectively. We suppose now that the desired attitude is achieved when the initial frame $\mathbb{F}_1$ is aligned to the target frame $\mathbb{F}_2$; according to Euler’s theorem, this can be obtained by a single rotation of the frame $\mathbb{F}_1$ about an axis referred to as the Euler axis (or rotation eigenaxis), whose components do not depend on the particular reference frame ($\mathbb{F}_1$ or $\mathbb{F}_2$). Let $\boldsymbol{e} \in \Re^3$ represent the Euler axis unit vector, and let $\phi \in (0, \pi)$ represent the Euler angle of rotation about the Euler axis.

Euler axis and angle can be expressed as a function of the entries of the rotation matrix as follows [1]:

$$\cos\phi = \frac{1}{2}[\mathbb{T}_{21}(1,1) + \mathbb{T}_{21}(2,2) + \mathbb{T}_{21}(3,3) - 1] \tag{2}$$

$$\sin\phi = \pm\sqrt{1-\cos^2\phi} \tag{3}$$

and

$$\boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2\sin\phi}\begin{bmatrix} \mathbb{T}_{21}(2,3) - \mathbb{T}_{21}(3,2) \\ \mathbb{T}_{21}(3,1) - \mathbb{T}_{21}(1,3) \\ \mathbb{T}_{21}(1,2) - \mathbb{T}_{21}(2,1) \end{bmatrix} \tag{4}$$

whereas the reciprocal equations lead to an expression of the rotation matrix in terms of Euler axis/angle:

$$\mathbb{T}_{21} = \cos\phi\boldsymbol{I} + (1-\cos\phi)\boldsymbol{e}\cdot\boldsymbol{e}^T - \sin\phi(\boldsymbol{e}\times) \tag{5}$$

where $\boldsymbol{e}\times$ is the skew symmetric cross-product operator defined as follows:

$$\boldsymbol{e}\times = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \tag{6}$$

and $\boldsymbol{I}$ is the 3×3 unity matrix.

III. Problem Statement and Solution

Given a generic desired attitude achievable by a rotation $\phi \neq 0, \pi$ about the eigenaxis $\boldsymbol{e}$ (the two cases $\phi = 0, \pi$ represent singularities in the Euler’s theorem, see Eq. (4), and therefore must be excluded), there could be cases where the required rotation cannot be performed. Let $\boldsymbol{g} \in \Re^3$ be a unit vector not aligned to the Euler axis $\boldsymbol{e}$: a rotation about $\boldsymbol{g}$ by the angle ϕ, would take the initial frame $\mathbb{F}_1$ to a frame $\mathbb{F}_3$ which is not aligned to the target frame $\mathbb{F}_2$.

Next, we develop an analytical expression for the particular rotation angle $\hat{\phi} \in [0, \pi)$, about a generic axis $\boldsymbol{g}$ not aligned to $\boldsymbol{e}$ (in

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what follows, g is considered as given and no optimization is performed on it), which minimizes the alignment error between the target frame $\mathbb{F}_2$ and the attainable frame $\mathbb{F}_3$.

According to Eq. (5), the rotation matrix which takes $\mathbb{F}_1$ to $\mathbb{F}_3$, expressed in terms of the Euler axis/angle, is given by

$$\mathbb{T}_{31} = \cos\hat{\phi}\boldsymbol{I} + (1-\cos\hat{\phi})\boldsymbol{g}\cdot\boldsymbol{g}^T - \sin\hat{\phi}(\boldsymbol{g}\times) \tag{7}$$

where $\boldsymbol{g}\times$ is the skew symmetric matrix defined in Eq. (6). Thus the rotation matrix which takes $\mathbb{F}_3$ to $\mathbb{F}_2$ is given by

$$\mathbb{T}_{23} = \mathbb{T}_{21}\mathbb{T}_{13} = \mathbb{T}_{21}\mathbb{T}_{31}^T \tag{8}$$

On the basis of Eq. (2), the alignment error is derived as follows:

$$\cos\epsilon = \frac{1}{2}[\mathbb{T}_{23}(1,1) + \mathbb{T}_{23}(2,2) + \mathbb{T}_{23}(3,3) - 1] \tag{9}$$

By using Eqs. (5) and (7) in Eq. (8), and after some simplifications shown in the Appendix, Eq. (9) can be written in the following compact form:

$$\cos\epsilon = \frac{1}{2}\{(\boldsymbol{e}\cdot\boldsymbol{g})[(\boldsymbol{e}\cdot\boldsymbol{g})(1-\cos\phi)(1-\cos\hat{\phi}) + 2\sin\hat{\phi}\sin\phi] + (\cos\phi+1)\cos\hat{\phi} + (\cos\phi - 1)\} \tag{10}$$

Similar to Eq. (3) we have

$$\sin\hat{\phi} = \pm\sqrt{1-\cos^2\hat{\phi}} \tag{11}$$

so that Eq. (10) becomes

$$\cos\epsilon = \frac{1}{2}\{(\boldsymbol{e}\cdot\boldsymbol{g})[(\boldsymbol{e}\cdot\boldsymbol{g})(1-\cos\phi)(1-\cos\hat{\phi}) \pm 2\sin\phi\sqrt{1-\cos^2\hat{\phi}}] + (\cos\phi+1)\cos\hat{\phi} + (\cos\phi-1)\} \tag{12}$$

To minimize the alignment error, Eq. (12) is differentiated with respect to the variable $\cos\hat{\phi}$, leading to

$$\frac{d\cos\epsilon}{d\cos\hat{\phi}} = f_\phi \mp g_\phi\frac{\cos\hat{\phi}}{\sqrt{1-\cos^2\hat{\phi}}} \tag{13}$$

where

$$f_\phi = \frac{1}{2}[(\boldsymbol{e}\cdot\boldsymbol{g})^2(\cos\phi - 1) + (\cos\phi + 1)] \tag{14}$$

and

$$g_\phi = (\boldsymbol{e}\cdot\boldsymbol{g})\sin\phi \tag{15}$$

Without any loss of generality, we assume that $\sin\phi > 0$ (i.e., the positive solution of Eq. (3) is selected) and $\boldsymbol{g}$ is selected so that $\boldsymbol{e}\cdot\boldsymbol{g}\geq 0$.

To minimize the alignment error angle ϵ, its cosine must be maximized; thus, setting Eq. (13) to zero and solving for $\cos\hat{\phi}$, one has

$$f_\phi\sqrt{1-\cos^2\hat{\phi}} = \pm g_\phi\cos\hat{\phi} \tag{16}$$

Let us now consider the second derivative:

$$\frac{d^2\cos\epsilon}{d\cos^2\hat{\phi}} = \mp g_\phi\frac{\sqrt{1-\cos^2\hat{\phi}} + 2\cos^2\hat{\phi}/\sqrt{1-\cos^2\hat{\phi}}}{1-\cos^2\hat{\phi}} \tag{17}$$

Because the function g_ϕ is always nonnegative, it is easy to notice that Eq. (17) is negative, thus leading to the maximization of $\cos\epsilon$ when the positive solution of Eq. (11) is selected. Thus, by imposing $\sin\hat{\phi} > 0$, Eq. (16) becomes

$$f_\phi\sqrt{1-\cos^2\hat{\phi}} = g_\phi\cos\hat{\phi} \tag{18}$$

Because $\sqrt{1-\cos^2\hat{\phi}}$ and g_ϕ are nonnegative terms, Eq. (18) gives the result that f_ϕ and $\cos\hat{\phi}$ must have the same sign.

By solving Eq. (18), one has

$$\cos\hat{\phi} = \pm\frac{f_\phi}{\sqrt{f_\phi^2 + g_\phi^2}} \tag{25}$$

and the positive solution is retained for what concluded after Eq. (18). This leads to

$$\hat{\phi} = \arccos\frac{f_\phi}{\sqrt{f_\phi^2 + g_\phi^2}} \tag{20}$$

with $0 \leq \hat{\phi} < \pi$.

IV. Numerical Results

To verify the analytical result presented in Eq. (20), numerical simulations were performed. First, we assumed a certain (known) transformation between the initial reference frame $\mathbb{F}_1$ and the target reference frame $\mathbb{F}_2$ with a nominal Euler axis $\boldsymbol{e}$ and a nominal Euler angle ϕ. Then, we computed the attitude error ϵ for some possible values of the scalar product $\boldsymbol{e}\cdot\boldsymbol{g}$ (0, 0.25, 0.45, 0.6, 0.75, 0.9, and 1) and for $\hat{\phi}$ in the range $(0,\pi)$. Correspondingly, we computed the value of the nonnominal Euler angle $\hat{\phi}$ which, according to Eq. (20), should guarantee a minimum attitude error ϵ for a given value of $\boldsymbol{e}\cdot\boldsymbol{g}$ and ϕ. The result is shown in Fig. 1 for $\phi = 70$ deg. For $\boldsymbol{e}\cdot\boldsymbol{g} = 0$ (upper curve) a rotation angle $\hat{\phi} = 0$ guarantees the minimum attitude error (i.e., there is nothing which can be done to improve the alignment between the initial and the target reference frames) which remains equal to the initial Euler angle ϕ. For increasing values of the scalar product $\boldsymbol{e}\cdot\boldsymbol{g}$, the situation improves. The minimum on each curve is reached for values of $\hat{\phi}\neq 0$, thus showing that a rotation about $\boldsymbol{g}$, even if not aligned to $\boldsymbol{e}$, yields a reduction of the alignment error ϵ. For $\boldsymbol{e}\cdot\boldsymbol{g} = 1$ ($\boldsymbol{e}\equiv\boldsymbol{g}$), the optimal rotation angle $\hat{\phi}\equiv\phi = 70$ deg and, correspondingly, the attitude error ϵ reaches the zero value, as expected.

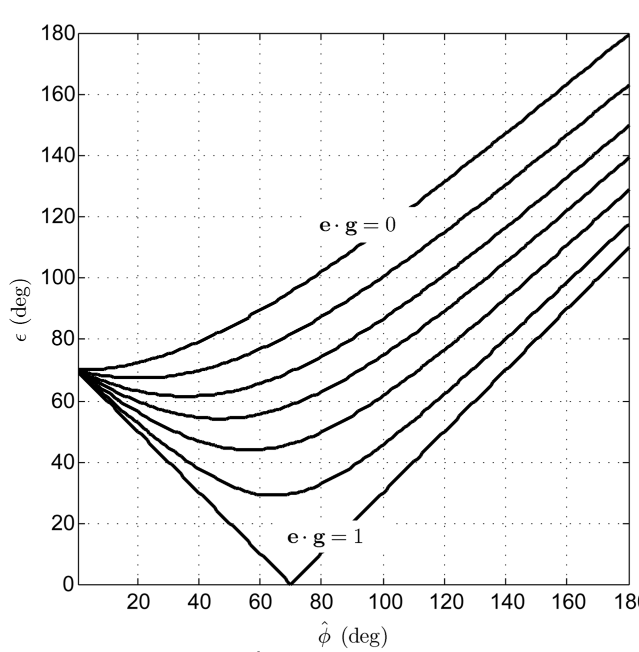

Fig. 1 Attitude error ϵ vs $\hat{\phi}$ for $\phi = 70$ deg for different values of $\boldsymbol{e}\cdot\boldsymbol{g}$.

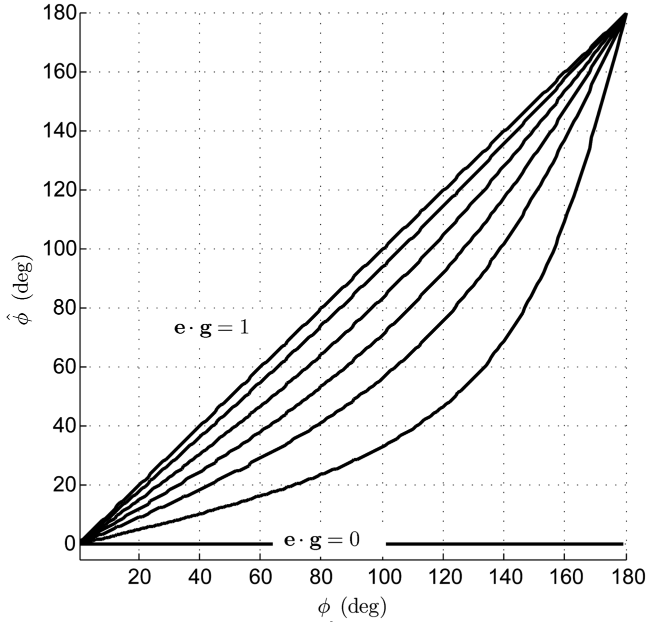

Fig. 2 Nonnominal Euler angle $\hat{\phi}$ vs ϕ for different values of $e \cdot g$.

Another way to check the correctness of Eq. (20) is to observe (see Fig. 2) the behavior of the nonnominal Euler angle $\hat{\phi}$, as a function of the Euler angle ϕ, for the same values of the scalar product $\boldsymbol{e} \cdot \boldsymbol{g}$ already shown in Fig. 1. Again, for $\boldsymbol{e} \cdot \boldsymbol{g} = 0$, the optimal nonnominal Euler angle $\hat{\phi}$ computed by Eq. (20) is zero for any value of the Euler angle ϕ. This indicates that for $\boldsymbol{e} \perp \boldsymbol{g}$ no possible rotations around the $\boldsymbol{g}$ axis can reduce the attitude error ϵ. When $\boldsymbol{e} \cdot \boldsymbol{g}$ increases (i.e., the $\boldsymbol{g}$ axis gets closer to the $\boldsymbol{e}$ axis), the optimal nonnominal Euler angle $\hat{\phi}$ starts increasing, indicating that a rotation of an angle $\hat{\phi}$ about the $\boldsymbol{g}$ axis reduces the attitude error ϵ. Finally, for $\boldsymbol{e} \cdot \boldsymbol{g} = 1$, the optimal nonnominal Euler angle $\hat{\phi}$ is always identical to ϕ, as already shown in Fig. 1. It is worth noticing that $\hat{\phi} \le \phi$, for each ϕ and all values of $\boldsymbol{e} \cdot \boldsymbol{g}$.

VI. Conclusions

Assuming that a reference frame $\mathbb{F}_1$ can be rotated about an Euler axis $\boldsymbol{e}$ by the Euler angle ϕ to reach the target reference frame $\mathbb{F}_2$, we have presented an analytical method to compute the rotation angle $\hat{\phi}$ by which $\mathbb{F}_1$ must be rotated about a different (nonnominal) Euler axis $\boldsymbol{g}$ to minimize the attitude error between the attained reference frame $\mathbb{F}_3$ and the target frame $\mathbb{F}_2$. This procedure makes use only of the hypothesis that, given a certain Euler axis $\boldsymbol{e}$, the direction of the nonnominal axis $\boldsymbol{g}$ is selected so that $\boldsymbol{e} \cdot \boldsymbol{g} \ge 0$ and, given a $0 < \phi < \pi$, also $0 \le \hat{\phi} < \pi$. The proposed procedure can be useful when applied within attitude control problems where a desired rotation is not attainable. Examples of this class of problems are represented by underactuated S/C, owing to the possible lack of actuators capable of rotating the S/C around a certain axis, and magnetically controlled S/C, because of the variable direction, in body axes, of the external magnetic field.

Appendix: Alignment error expression

$$
\begin{aligned}
\cos\epsilon &= \frac{1}{2}[\mathbb{T}_{23}(1,1) + \mathbb{T}_{23}(2,2) + \mathbb{T}_{23}(3,3) - 1] \\
&= g_1 g_3 e_1 e_3 + g_2 g_3 e_2 e_3 + \frac{g_1^2 e_1^2}{2} + g_2 g_3 \cos\hat{\phi} e_2 e_3 \cos\phi \\
&- g_2 g_3 \cos\hat{\phi} e_2 e_3 - \frac{1}{2}\cos\hat{\phi} e_2^2 \cos\phi - \frac{1}{2} g_3^2 \cos\hat{\phi} e_3^2 \\
&- \frac{1}{2} g_2^2 \cos\hat{\phi} e_2^2 - \frac{1}{2} g_2^2 \cos\hat{\phi}\cos\phi + g_1 g_3 \cos\hat{\phi} e_1 e_3 \cos\phi \\
&- g_1 g_3 \cos\hat{\phi} e_1 e_3 + g_1 g_2 \cos\hat{\phi} e_1 e_2 \cos\phi - g_1 g_2 \cos\hat{\phi} e_1 e_2 \\
&+ \frac{1}{2}\cos\hat{\phi} e_3^2 + \frac{3}{2}\cos\hat{\phi}\cos\phi + \frac{1}{2}\cos\hat{\phi} e_2^2 - \frac{1}{2}\cos\hat{\phi} e_3^2 \cos\phi \\
&- \frac{1}{2} g_3^2 \cos\hat{\phi}\cos\phi + \frac{1}{2} g_2^2 \cos\hat{\phi} e_2^2 \cos\phi + \frac{1}{2}\cos\hat{\phi} e_1^2 \\
&- \frac{1}{2} g_1^2 \cos\hat{\phi} e_1^2 - \frac{1}{2}\cos\hat{\phi} e_1^2 \cos\phi - \frac{1}{2} g_1^2 \cos\hat{\phi}\cos\phi \\
&+ \sin\hat{\phi} g_1 \sin\phi e_1 + \frac{1}{2} g_3^2 \cos\hat{\phi} e_3^2 \cos\phi + \sin\hat{\phi} g_3 \sin\phi e_3 \\
&+ \sin\hat{\phi} g_2 \sin\phi e_2 - \frac{1}{2} + g_1 g_2 e_1 e_2 + \frac{g_3^2 e_3^2}{2} + \frac{g_2^2 e_2^2}{2} \\
&- \frac{1}{2} g_1^2 e_1^2 \cos\phi - \frac{1}{2} g_3^2 e_3^2 \cos\phi - \frac{1}{2} g_2^2 e_2^2 \cos\phi + \frac{1}{2} g_3^2 \cos\phi \\
&+ \frac{1}{2} g_1^2 \cos\phi + \frac{1}{2} g_2^2 \cos\phi + \frac{1}{2} g_1^2 \cos\hat{\phi} e_1^2 \cos\phi \\
&- g_1 g_3 e_1 e_3 \cos\phi - g_1 g_2 e_1 e_2 \cos\phi - g_2 g_3 e_2 e_3 \cos\phi \qquad \text{(A1)}
\end{aligned}
$$

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